*Journal of Statistical Physics, Vol. 78, Nos. 5/6, 1995*

# Approximation of the Boltzmann Equation by Discrete Velocity Models

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*Received May 3, 1994; final August 30, 1994*

Two convergence results related to the approximation of the Boltzmann equation by discrete velocity models are presented. First we construct a sequence of deterministic discrete velocity models and prove convergence (as the number of discrete velocities tends to infinity) of their solutions to the solution of a spatially homogeneous Boltzmann equation. Second we introduce a sequence of Markov jump processes (interpreted as random discrete velocity models) and prove convergence (as the intensity of jumps tends to infinity) of these processes to the solution of a deterministic discrete velocity model.

**KEY WORDS**: Boltzmann equation; discrete velocity models; weak convergence; random mass flow.

## 1. INTRODUCTION

We consider the spatially homogeneous Boltzmann equation (cf. ref. 6, p. 26, or ref. 3, p. 392)

$$\frac{\partial}{\partial t} p(t, v) = \int_{\mathscr{R}^3} dw \int_{\mathscr{S}^2} de\, B(v, w, e)[p(t, v^*)\, p(t, w^*) - p(t, v)\, p(t, w)] \quad (1.1)$$

with the initial condition

$$p(0, v) = p_0(v) \quad (1.2)$$

The symbols $dw$ and $de$ denote the Lebesgue measure on the three-dimensional Euclidean space $\mathscr{R}^3$ and the uniform surface measure on the unit sphere $\mathscr{S}^2$, respectively. The function $B$ is called the collision kernel. The objects $v^*$ and $w^*$ are defined as

$$v^* = v + e(e, w - v), \qquad w^* = w + e(e, v - w)$$

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0022-4715/95/0300-1555$07.50/0 

where $v, w \in \mathscr{R}^3$, $e \in \mathscr{S}^2$, and $(\cdot, \cdot)$ denotes the scalar product. They are interpreted as the postcollision velocities of two particles with the precollision velocities $v$ and $w$. Equations (1.1)–(1.2) describe the time evolution of a probability density function $p(t, v)$ on the velocity space $\mathscr{R}^3$.

The purpose of this paper is to study the problem of approximating the measures

$$\lambda(t, dv) = p(t, v)\, dv$$

which correspond to the solution of the Boltzmann equation (1.1)–(1.2), by measures concentrated on a finite subset of the velocity space.

The investigation will be carried out in the following more general setup. Let $(\mathscr{Z}, r)$ be a locally compact, separable metric space ($r$ denoting the metric) and $\mathscr{B}_{\mathscr{Z}}$ denote the Borel $\sigma$-algebra. Let $B(\mathscr{Z})$ be the Banach space of bounded Borel measurable functions on $\mathscr{Z}$ with $\|\varphi\| = \sup_{z \in \mathscr{Z}} |\varphi(z)|$, and let $\bar{C}(\mathscr{Z})$ denote the subspace of bounded continuous functions. Furthermore, let $\mathscr{M}(\mathscr{Z})$ be the space of finite, positive measures on $(\mathscr{Z}, \mathscr{B}_{\mathscr{Z}})$.

Let $Q$ be a function on $\mathscr{Z} \times \mathscr{Z} \times \mathscr{B}_{\mathscr{Z}} \times \mathscr{B}_{\mathscr{Z}}$ with the properties

$$Q(z_1, z_2, \cdot, \Gamma), Q(z_1, z_2, \Gamma, \cdot) \in \mathscr{M}(\mathscr{Z}), \qquad \forall z_1, z_2 \in \mathscr{Z}, \quad \Gamma \in \mathscr{B}_{\mathscr{Z}} \quad (1.3)$$

$$Q(\cdot, \cdot, \Gamma_1, \Gamma_2) \in B(\mathscr{Z} \times \mathscr{Z}), \qquad \forall \Gamma_1, \Gamma_2 \in \mathscr{B}_{\mathscr{Z}} \quad (1.4)$$

and

$$Q(z_1, z_2, \mathscr{Z}, \mathscr{Z}) \leqslant C_{Q,\max}, \qquad \forall z_1, z_2 \in \mathscr{Z} \quad (1.5)$$

We consider the equation

$$\frac{d}{dt} \int_{\mathscr{Z}} \varphi(z)\, \lambda(t, dz) = \int_{\mathscr{Z}} \int_{\mathscr{Z}} \left\{ \int_{\mathscr{Z}} \int_{\mathscr{Z}} [\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2) - \varphi(z_1) - \varphi(z_2)] \, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \right\} \lambda(t, dz_1)\, \lambda(t, dz_2), \qquad \lambda(0) = \lambda_0 \quad (1.6)$$

where $\varphi \in B(\mathscr{Z})$ and $\lambda_0 \in \mathscr{M}(\mathscr{Z})$. The solution $\lambda(t)$ is a function of the time variable $t \in [0, \infty)$ taking values in $\mathscr{M}(\mathscr{Z})$.

First we note that Eq. (1.6) is a generalized weak form of the Boltzmann equation (1.1)–(1.2).[12,11] To see that, one considers $\mathscr{Z} = \mathscr{R}^3$, and the collision kernel $Q$ of the form

$$Q(v_1, v_2, \Gamma_1, \Gamma_2) = \int_{\mathscr{S}^2} \tfrac{1}{2} B(v_1, v_2, e)\, \delta_{v_1^*}(\Gamma_1)\, \delta_{v_2^*}(\Gamma_2)\, de \quad (1.7)$$

where

$$v_1^* = v_1 + e(e, v_2 - v_1), \qquad v_2^* = v_2 + e(e, v_1 - v_2) \tag{1.8}$$

and $\delta_z$ denotes the Dirac measure concentrated in $z$. The integral with respect to $Q$ on the right-hand side of Eq. (1.6) takes the form

$$\int_{\mathscr{S}^2} \tfrac{1}{2} B(v_1, v_2, e)[\varphi(v_1^*) + \varphi(v_2^*) - \varphi(v_1) - \varphi(v_2)]\, de$$

After the substitution of the integration variables $(v_1, v_2)$ by $(v_1^*, v_2^*)$ and removing the test function $\varphi$, one obtains Eqs. (1.1)–(1.2) provided that the kernel $B$ has the properties

$$B(v_1, v_2, e) = B(v_2, v_1, e) = B(v_1^*, v_2^*, e)$$

Next we consider Eq. (1.6) in the case $\mathscr{Z} = \mathscr{Z}^{(N)}$, where

$$\mathscr{Z}^{(N)} = \{\xi_1^{(N)},..., \xi_N^{(N)}\}, \qquad \xi_i^{(N)} \in \mathscr{R}^3, \quad i = 1,..., N$$

The solution $\lambda^{(N)}(t)$ is determined by its values $\lambda^{(N)}(t, \{\xi_i^{(N)}\}) =: p_i^{(N)}(t)$, $i = 1,..., N$. Considering the functions

$$\varphi_m(v) = ¶_{\{\xi_m^{(N)}\}}(v), \qquad m = 1,..., N$$

where $¶_\Gamma$ denotes the indicator function of a set $\Gamma$, shows that Eq. (1.6) is equivalent to the system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt} p_m^{(N)}(t) = \sum_{i,j,k,l=1}^{N} & [\varphi_m(\xi_k^{(N)}) + \varphi_m(\xi_l^{(N)}) - \varphi_m(\xi_i^{(N)}) - \varphi_m(\xi_j^{(N)})] \\ & \times Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\})\, p_i^{(N)}(t)\, p_j^{(N)}(t) \end{aligned} \tag{1.9}$$

$$p_m^{(N)}(0) = \lambda_0^{(N)}(\{\xi_m^{(N)}\}), \qquad m = 1,..., N \tag{1.10}$$

Moreover, one easily realizes that the system (1.9)–(1.10) is equivalent to the system

$$\begin{aligned} & \frac{d}{dt} p_i^{(N)}(t) \\ & = \sum_{j,k,l=1}^{N} \{\alpha^{(N)}(k, l, i, j)\, p_k^{(N)}(t)\, p_l^{(N)}(t) - \alpha^{(N)}(i, j, k, l)\, p_i^{(N)}(t)\, p_j^{(N)}(t)\} \end{aligned} \tag{1.11}$$

$$p_i^{(N)}(0) = \lambda_0^{(N)}(\{\xi_i^{(N)}\}), \qquad i = 1,..., N \tag{1.12}$$

where

$$\alpha^{(N)}(i, j, k, l)$$
$$= \tfrac{1}{2}[Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\}) + Q^{(N)}(\xi_j^{(N)}, \xi_i^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\})$$
$$+ Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_l^{(N)}\}, \{\xi_k^{(N)}\}) + Q^{(N)}(\xi_j^{(N)}, \xi_i^{(N)}, \{\xi_l^{(N)}\}, \{\xi_k^{(N)}\})] \quad (1.13)$$

The system (1.11)–(1.12) is called a discrete velocity model (in the spatially homogeneous case). It describes the time evolution of the weight functions $p_i^{(N)}(t)$, which correspond to the "discrete velocities" $\xi_i^{(N)}$, $i = 1,..., N$. We refer to refs. 10 and 12 concerning more details about discrete velocity models.

In Section 2 we are concerned with the problem of approximating the solution $\lambda(t)$ of the generalized Boltzmann equation (1.6) by measures of the form

$$\lambda^{(N)}(t) = \sum_{i=1}^{N} p_i^{(N)}(t)\, \delta_{\xi_i^{(N)}}$$

where $\delta$ in the Dirac measure, and $p_i^{(N)}(t)$, $i = 1,..., N$, is the solution to a discrete velocity model of the form (1.11)–(1.13). We construct an appropriate sequence $(\tilde{\mathscr{Z}}^{(N)}, \tilde{\lambda}_0^{(N)}, \tilde{Q}^{(N)})$ and prove that

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \varrho(\lambda(t), \tilde{\lambda}^{(N)}(t)) = 0, \qquad \forall T > 0$$

where $\varrho$ is a metric, which is equivalent to weak convergence in the space $\mathscr{M}(\mathscr{Z})$.

In Section 3 we study the problem of approximating the solution $p_i^{(N)}(t)$, $i = 1,..., N$, of a discrete velocity model (1.11)–(1.13) by a stochastic process. We introduce a Markov jump process $(g_i^{(N,\gamma)}(t))_{i=1}^N$, where $\gamma$ is a parameter governing the intensity of the jumps. The functions $g_i^{(N,\gamma)}(t)$ are interpreted as random weight functions, which correspond to the discrete velocities $\xi_i^{(N)}$, $i = 1,..., N$. A model of such type was introduced in ref. 5 and called a random discrete velocity model. We prove that

$$\lim_{\gamma\to\infty} E \sup_{0\leq t\leq T} \sum_{i=1}^{N} |p_i^{(N)}(t) - g_i^{(N,\gamma)}(t)| = 0, \qquad \forall T > 0$$

where $E$ denotes mathematical expectation.

## 2. A CONVERGENCE RESULT FOR DISCRETE VELOCITY MODELS

Let $\{\Delta_1^{(N)},..., \Delta_N^{(N)}\}$ be a measurable partition of the space $\mathscr{X}$ into disjoint cells, i.e.,

$$\Delta_i^{(N)} \in \mathscr{B}_{\mathscr{X}}, \qquad \Delta_i^{(N)} \cap \Delta_j^{(N)} = \varnothing, \qquad i, j = 1,..., N, \quad i \neq j$$

where $\varnothing$ denotes the empty set, and

$$\bigcup_{i=1}^{N} \Delta_i^{(N)} = \mathscr{X}, \qquad \forall N = 1, 2,...$$

Let $\xi_i^{(N)} \in \Delta_i^{(N)}$, $i = 1,..., N$, and consider the space

$$\tilde{\mathscr{X}}^{(N)} = \{\xi_1^{(N)},..., \xi_N^{(N)}\} \tag{2.1}$$

with the discrete topology. Define the transformation $I^{(N)}$ as

$$I^{(N)}(\mu) = \sum_{i=1}^{N} \mu(\Delta_i^{(N)})\, \delta_{\xi_i^{(N)}}, \qquad \mu \in \mathscr{M}(\mathscr{X}) \tag{2.2}$$

We consider $I^{(N)}(\mu)$ as a measure on $\tilde{\mathscr{X}}^{(N)}$ as well as a measure on $\mathscr{X}$, denoting it with the same symbol. Define

$$\tilde{\lambda}_0^{(N)} = I^{(N)}(\lambda_0) \tag{2.3}$$

and

$$\tilde{Q}^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\}) = Q(\xi_i^{(N)}, \xi_j^{(N)}, \Delta_k^{(N)}, \Delta_l^{(N)}) \tag{2.4}$$

where $\lambda_0$ and $Q$ are the initial value and the collision kernel, respectively, of the Boltzmann equation (1.6).

On $\mathscr{M}(\mathscr{X})$, we consider the bounded Lipschitz metric (cf. ref. 4, p. 150)

$$\varrho(\mu_1, \mu_2) = \sup_{\varphi \in B(\mathscr{X}): \|\varphi\|_L \leq 1} |\langle \varphi, \mu_1 \rangle| - \langle \varphi, \mu_2 \rangle|, \qquad \mu_1, \mu_2 \in \mathscr{M}(\mathscr{X}) \tag{2.5}$$

where

$$\|\varphi\|_L = \max\left( \sup_{x \in \mathscr{X}} |\varphi(x)|, \sup_{x, y \in \mathscr{X}, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{r(x, y)} \right) \tag{2.6}$$

and

$$\langle \varphi, \mu \rangle = \int_{\mathscr{X}} \varphi(y)\, \mu(dy), \qquad \varphi \in B(\mathscr{X}), \quad \mu \in \mathscr{M}(\mathscr{X}) \tag{2.7}$$

**Theorem 2.1**. Let $p_i^{(N)}(t)$, $i=1,\dots,N$, be the solution of the discrete velocity model (1.11)–(1.13) corresponding to $\tilde{\mathscr{Z}}^{(N)}$, $\tilde{\lambda}_0^{(N)}$, and $\tilde{Q}^{(N)}$ defined in (2.1), (2.3), and (2.4), and let $\tilde{\lambda}^{(N)}(t)$ be the measure-valued function defined as

$$\tilde{\lambda}^{(N)}(t)=\sum_{i=1}^{N} p_i^{(N)}(t)\,\delta_{\xi_i^{(N)}}$$

Suppose

$$\lim_{N\to\infty}\ \max_{i:\Delta_i^{(N)}\cap K\neq\varnothing}\ \operatorname{diam}(\Delta_i^{(N)})=0, \qquad \forall \text{ compact } K\subset\mathscr{Z} \tag{2.8}$$

where $\operatorname{diam}(\Gamma)=\sup_{x,y\in\Gamma} r(x,y)$, $\Gamma\subset\mathscr{Z}$.

Let the kernel $Q$ satisfy (1.3)–(1.5) and

$$\int_{\mathscr{Z}}\int_{\mathscr{Z}} [\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)]\,Q(\cdot,\cdot,d\tilde{z}_1,dz_2)\in\bar{C}(\mathscr{Z}\times\mathscr{Z}), \qquad \forall\varphi\in\bar{C}(\mathscr{Z}) \tag{2.9}$$

Then

$$\lim_{N\to\infty}\ \sup_{0\leqslant t\leqslant T}\ \varrho(\lambda(t),\tilde{\lambda}^{(N)}(t))=0, \qquad \forall T>0$$

where $\lambda(t)$ is the solution of the Boltzmann equation (1.6).

**Remark 2.2**. Consider the particular case (1.7), (1.8). Assumptions (1.3)–(1.5) and (2.9) are fulfilled if the function $B(v_1,v_2,e)$ is continuous in $(v_1,v_2)$ and satisfies

$$B(v_1,v_2,e)\leqslant B_{\max}(e), \qquad \forall v_1,v_2\in\mathscr{R}^3, \qquad \forall e\in\mathscr{S}^2$$

where $\int_{\mathscr{S}^2} B_{\max}(e)\,de<\infty$. Thus, a truncation of the collision kernel is necessary in most applications [e.g., $B(v_1,v_2,e)=|(e,v_2-v_1)|$, in the hard-sphere case].

To prepare the proof of Theorem 2.1, we describe the representation of the solution of Eq. (1.6) in form of Wild's sum.[14,8,11,1] Using assumption (1.5), we introduce a kernel

$$\begin{aligned} &Q_{\max}(z_1,z_2,\Gamma_1,\Gamma_2)\\ &\quad=Q(z_1,z_2,\Gamma_1,\Gamma_2)+[C_{Q,\max}-Q(z_1,z_2,\mathscr{Z},\mathscr{Z})]\,\delta_{z_1}(\Gamma_1)\,\delta_{z_2}(\Gamma_2)\\ &\qquad\qquad z_1,z_2\in\mathscr{Z}, \quad \Gamma_1,\Gamma_2\in\mathscr{B}_{\mathscr{Z}} \end{aligned} \tag{2.10}$$

Furthermore, we define an operator $K_{\max}: \mathcal{M}(\mathcal{Z}) \times \mathcal{M}(\mathcal{Z}) \to \mathcal{M}(\mathcal{Z})$ as

$$K_{\max}(\mu_1, \mu_2)(\Gamma) = \int_{\mathcal{Z}} \int_{\mathcal{Z}} [Q_{\max}(z_1, z_2, \Gamma, \mathcal{Z}) + Q_{\max}(z_1, z_2, \mathcal{Z}, \Gamma)] \, \mu_1(dz_1) \, \mu_2(dz_2)$$

$$\mu_1, \mu_2 \in \mathcal{M}(\mathcal{Z}), \quad \Gamma \in \mathcal{B}_{\mathcal{Z}} \tag{2.11}$$

We note that, with the above notations, Eq. (1.6) takes the form

$$\frac{d}{dt} \langle \varphi, \lambda(t) \rangle = \langle \varphi, K_{\max}(\lambda(t), \lambda(t)) \rangle - 2C_{Q,\max} \lambda_0(\mathcal{Z}) \langle \varphi, \lambda(t) \rangle, \quad \lambda(0) = \lambda_0$$

It is easy to check that there is a unique solution of Eq. (1.6). This solution is represented in the form

$$\lambda(t) = \sum_{k=1}^{\infty} e^{-c_0 t}(1 - e^{-c_0 t})^{k-1} \nu_k, \qquad t \in [0, \infty) \tag{2.12}$$

where

$$\nu_1 = \lambda_0, \qquad \nu_{k+1} = \frac{1}{c_0 k} \sum_{i=1}^{k} K_{\max}(\nu_i, \nu_{k+1-i}), \qquad k \geq 1 \tag{2.13}$$

and

$$c_0 = 2C_{Q,\max} \lambda_0(\mathcal{Z}) \tag{2.14}$$

We assume $\lambda_0(\mathcal{Z}) > 0$, to avoid trivialities. One easily shows by induction on $k$ that

$$\nu_k(\mathcal{Z}) = \lambda_0(\mathcal{Z}), \qquad \forall k = 1, 2, \ldots \tag{2.15}$$

The series (2.12) converges in the total variation norm.

The Wild sum representation (2.12)–(2.14) shows rather explicitly how the solution $\lambda(t)$ depends on the objects $\mathcal{Z}$, $\lambda_0$, and $Q$ that determine Eq. (1.6). First we study the stability of the solution with respect to these objects.

Let $(\mathcal{Z}^{(N)})$ be a sequence of subspaces of $\mathcal{Z}$ endowed with the relative topology. Note that any measure $\mu$ on $\mathcal{Z}^{(N)}$ has a natural extension $\hat{\mu}$ on $\mathcal{Z}$ defined as

$$\hat{\mu}(\Gamma) = \mu(\Gamma \cap \mathcal{Z}^{(N)}), \qquad \Gamma \in \mathcal{B}_{\mathcal{Z}}$$

Let, for $N=1,2,\ldots$, $\lambda_0^{(N)}\in\mathcal{M}(\mathcal{Z}^{(N)})$ and $Q^{(N)}$ be a kernel having the properties (1.3), (1.4) with $\mathcal{Z}$ replaced by $\mathcal{Z}^{(N)}$. Assume that (1.5) holds uniformly in $N$, namely

$$Q^{(N)}(z_1,z_2,\mathcal{Z}^{(N)},\mathcal{Z}^{(N)})\leqslant C_{Q,\max},\qquad \forall z_1,z_2\in\mathcal{Z}^{(N)},\qquad \forall N=1,2,\ldots \tag{2.16}$$

Let $K_{\max}^{(N)}$ be defined in analogy with (2.11), and let $\hat{K}_{\max}^{(N)}(\mu_1^{(N)},\mu_2^{(N)})$ denote the extension of the measure $K_{\max}^{(N)}(\mu_1^{(N)},\mu_2^{(N)})$, where $\mu_1^{(N)},\mu_2^{(N)}\in\mathcal{M}(\mathcal{Z}^{(N)})$. Finally, let $\lambda^{(N)}(t)$ denote the solution of Eq. (1.6) corresponding to $(\mathcal{Z}^{(N)},\lambda_0^{(N)},Q^{(N)})$.

**Lemma 2.3.** Suppose (2.16) and

$$\lim_{N\to\infty}\varrho(\hat{K}_{\max}^{(N)}(\mu_1^{(N)},\mu_2^{(N)}),K_{\max}(\mu_1,\mu_2))=0 \tag{2.17}$$

for any sequences $\mu_1^{(N)},\mu_2^{(N)}\in\mathcal{M}(\mathcal{Z}^{(N)})$ and measures $\mu_1,\mu_2\in\mathcal{M}(\mathcal{Z})$ such that

$$\lim_{N\to\infty}\varrho(\hat{\mu}_i^{(N)},\mu_i)=0,\qquad i=1,2 \tag{2.18}$$

If

$$\lim_{N\to\infty}\varrho(\hat{\lambda}_0^{(N)},\lambda_0)=0 \tag{2.19}$$

then

$$\lim_{N\to\infty}\sup_{0\leqslant t\leqslant T}\varrho(\hat{\lambda}^{(N)}(t),\lambda(t))=0,\qquad \forall T>0$$

where $\lambda(t)$ is the solution of the Boltzmann equation (1.6).

*Proof.* Comparing the Wild sum representations (2.12)–(2.14) of $\lambda(t)$ and $\lambda^{(N)}(t)$, respectively, one obtains

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}&\varrho(\hat{\lambda}^{(N)}(t),\lambda(t))\\ &\leqslant\sum_{k=1}^{\infty}\sup_{0\leqslant t\leqslant T}\sup_{\|\varphi\|_L\leqslant 1}|e^{-c_0t}(1-e^{-c_0t})^{k-1}\langle\varphi,\nu_k\rangle\\ &\quad-e^{-c_0^{(N)}t}(1-e^{-c_0^{(N)}t})^{k-1}\langle\varphi,\hat{\nu}_k^{(N)}\rangle|\end{aligned} \tag{2.20}$$

where

$$\nu_1^{(N)}=\lambda_0^{(N)},\qquad \nu_{k+1}^{(N)}=\frac{1}{c_0^{(N)}k}\sum_{i=1}^{k}K_{\max}^{(N)}(\nu_i^{(N)},\nu_{k+1-i}^{(N)}),\qquad k\geqslant 1 \tag{2.21}$$

and

$$c_0^{(N)} = 2C_{Q,\max}\lambda_0^{(N)}(\mathscr{Z}^{(N)}) \tag{2.22}$$

The series on the right-hand side of (2.20) has a majorant uniformly in $N$. This follows from (2.22), (2.15), and the fact that

$$\lim_{N\to\infty} \lambda_0^{(N)}(\mathscr{Z}^{(N)}) = \lambda_0(\mathscr{Z}) \tag{2.23}$$

which is a consequence of (2.19). Thus, it is sufficient to prove

$$\lim_{N\to\infty} s_k^{(N)} = 0, \qquad \forall k = 1, 2,...$$

where $s_k^{(N)}$ denotes the elements of the series on the right-hand side of (2.20). One obtains

$$\begin{aligned} s_k^{(N)} &\leqslant \varrho(\nu_k, \hat{\nu}_k^{(N)}) \\ &\quad + \sup_N \lambda_0^{(N)}(\mathscr{Z}^{(N)}) \sup_{0\leqslant t\leqslant T} |e^{-c_0 t}(1-e^{-c_0 t})^{k-1} - e^{-c_0^{(N)} t}(1-e^{-c_0^{(N)} t})^{k-1}| \end{aligned} \tag{2.24}$$

The second term on the right-hand side of (2.24) tends to zero as $N\to\infty$, since the sequence of functions $\exp(-c_0^{(N)}t)[1-\exp(-c_0^{(N)}t)]^{k-1}$ is equicontinuous on $[0, T]$ and tends to the function $\exp(-c_0 t)[1-\exp(-c_0 t)]^{k-1}$, for each $t\in[0, T]$, because of (2.22) and (2.23). Thus, it remains to show that

$$\lim_{N\to\infty} \varrho(\nu_k, \hat{\nu}_k^{(N)}) = 0, \qquad \forall k = 1, 2,... \tag{2.25}$$

This is done by induction on $k$. For $k=1$, (2.25) follows directly from (2.19). Considering the definitions (2.13) and (2.21) of $\nu_{k+1}$ and $\nu_{k+1}^{(N)}$, respectively, as well as (2.11), (2.15), and (2.14), shows that

$$\begin{aligned} \varrho(\nu_{k+1}, \hat{\nu}_{k+1}^{(N)}) &\leqslant \frac{1}{c_0 k} \sum_{i=1}^{k} \varrho(K_{\max}(\nu_i, \nu_{k+1-i}), \hat{K}_{\max}^{(N)}(\nu_i^{(N)}, \nu_{k+1-i}^{(N)})) \\ &\quad + \left|1 - \frac{c_0^{(N)}}{c_0}\right| \lambda_0^{(N)}(\mathscr{Z}^{(N)}) \end{aligned} \tag{2.26}$$

The second term on the right-hand side of (2.26) tends to zero as $N\to\infty$, because of (2.22) and (2.23). The first term tends to zero, because of the induction hypothesis and assumption (2.17), (2.18). ∎

The next two lemmas prepare the application of Lemma 2.3 to the special sequence $(\tilde{\mathscr{Z}}^{(N)}, \tilde{\lambda}_0^{(N)}, \tilde{Q}^{(N)})$ defined in (2.1), (2.3), (2.4).

**Lemma 2.4.** Let the kernel $Q$ satisfy (1.3)–(1.5) and (2.9).

Then the operator $K_{\max}$ defined in (2.11) is continuous with respect to weak convergence.

*Proof.* One obtains from (2.11) and (2.10) that

$$\begin{aligned}
&\langle \varphi, K_{\max}(\mu_1, \mu_2)\rangle \\
&= \int_{\mathscr{Z}} \int_{\mathscr{Z}} \left\{ \int_{\mathscr{Z}} \int_{\mathscr{Z}} [\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2)]\, Q_{\max}(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \right\} \mu_1(dz_1)\, \mu_2(z_2) \\
&= \int_{\mathscr{Z}} \int_{\mathscr{Z}} \left\{ \int_{\mathscr{Z}} \int_{\mathscr{Z}} [\varphi(\tilde{z}_1) + \varphi(\tilde{z}_2)]\, Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) \right. \\
&\quad \left. + [C_{Q,\max} - Q(z_1, z_2, \mathscr{Z}, \mathscr{Z})][\varphi(z_1) + \varphi(z_2)] \right\} \mu_1(dz_1)\, \mu_2(dz_2)
\end{aligned}$$

and the assertion follows. ∎

**Lemma 2.5.** Suppose assumption (2.8) is fulfilled. Let $\mu^{(N)}$, $\mu \in \mathscr{M}(\mathscr{Z})$ and $\lim_{N \to \infty} \varrho(\mu^{(N)}, \mu) = 0$. Then

$$\lim_{N \to \infty} \varrho(I^{(N)}(\mu^{(N)}), \mu) = 0$$

where $I^{(N)}$ is defined in (2.2).

*Proof.* The triangle inequality shows that it is sufficient to prove

$$\lim_{N \to \infty} \varrho(I^{(N)}(\mu^{(N)}), \mu^{(N)}) = 0 \tag{2.27}$$

Let $K \subset \mathscr{Z}$ be a compact set. Taking into account definitions (2.5)–(2.7), we obtain

$$\begin{aligned}
&\sup_{\|\varphi\|_L \leqslant 1} |\langle \varphi, I^{(N)}(\mu^{(N)})\rangle - \langle \varphi, \mu^{(N)}\rangle| \\
&\leqslant \sup_{\|\varphi\|_L \leqslant 1} \left| \sum_{i: \Delta_i^{(N)} \cap K \neq \varnothing} \left\{ \int_{\Delta_i^{(N)}} \varphi(z)\, I^{(N)}(\mu^{(N)})(dz) - \int_{\Delta_i^{(N)}} \varphi(z)\, \mu^{(N)}(dz) \right\} \right| \\
&\quad + \sup_{\|\varphi\|_L \leqslant 1} \left| \sum_{i: \Delta_i^{(N)} \cap K = \varnothing} \left\{ \int_{\Delta_i^{(N)}} \varphi(z)\, I^{(N)}(\mu^{(N)})(dz) - \int_{\Delta_i^{(N)}} \varphi(z)\, \mu^{(N)}(dz) \right\} \right|
\end{aligned}$$

$$\leqslant \sup_{\|\varphi\|_L \leqslant 1} \sum_{i:\Delta_i^{(N)} \cap K \neq \varnothing} \int_{\Delta_i^{(N)}} |\varphi(z) - \varphi(\xi_i^{(N)})| \, \mu^{(N)}(dz)$$

$$+ \sum_{i:\Delta_i^{(N)} \cap K = \varnothing} 2\mu^{(N)}(\Delta_i^{(N)})$$

$$\leqslant \left( \max_{i:\Delta_i^{(N)} \cap K \neq \varnothing} \operatorname{diam}(\Delta_i^{(N)}) \right) \mu^{(N)}(\mathscr{Z}) + 2\mu^{(N)}(\mathscr{Z}\backslash K) \tag{2.28}$$

Assertion (2.27) follows now from (2.28), assumption (2.8), and the tightness of the sequence $(\mu^{(N)})$. ▮

*Proof of Theorem 2.1.* In order to apply Lemma 2.3, we check conditions (2.16)–(2.19). Condition (2.16) follows directly from the definition (2.4) and assumption (1.5). Condition (2.19) is a particular case of Lemma 2.5, because of (2.3). Finally, we note that

$$K_{\max}^{(N)}(\mu_1^{(N)}, \mu_2^{(N)}) = I^{(N)}(K_{\max}(\mu_1^{(N)}, \mu_2^{(N)}))$$

where $\mu_1^{(N)}, \mu_2^{(N)} \in \mathscr{M}(\mathscr{Z}^{(N)})$, because of (2.4). Thus, condition (2.17), (2.18) follows from Lemma 2.5 provided that

$$\lim_{N \to \infty} \varrho(K_{\max}(\mu_1^{(N)}, \mu_2^{(N)}), K_{\max}(\mu_1, \mu_2)) = 0$$

if

$$\lim_{N \to \infty} \varrho(\mu_i^{(N)}, \mu_i) = 0, \qquad i = 1, 2$$

This is a consequence o Lemma 2.4 and assumptions (1.5) and (2.9). ▮

## 3. RANDOM MASS FLOW AND DISCRETE VELOCITY MODELS

We introduce a Markov process $Z(t) = (g_i^{(N,\gamma)}(t))_{i=1}^N$ with the state space $[0, C_{g,\max}]^N$, where $C_{g,\max} > 0$, and the infinitesimal generator

$$\mathscr{A}(\Phi)(\bar{z}) = \sum_{i,j,k,l=1}^{N} D^{(N,\gamma)}(\bar{z}, i, j, k, l)[\Phi(J^{(N,\gamma)}(\bar{z}, i, j, k, l)) - \Phi(\bar{z})] \tag{3.1}$$

where $\bar{z} = (g_1, ..., g_N)$, $\gamma \geqslant 1$ is a real number, and $\Phi$ is a bounded measurable function on the state space. The mapping $J^{(N,\gamma)}$ is a jump transformation defined as

$$[J^{(N,\gamma)}(\bar{z}, i, j, k, l)]_m = g_m + G^{(N,\gamma)}(\bar{z}, i, j, k, l)[\psi_{k,m} + \psi_{l,m} - \psi_{i,m} - \psi_{j,m}] \tag{3.2}$$

where

$$\psi_{i,m}=0, \qquad i\neq m, \qquad \psi_{i,i}=1, \quad i, m=1,..., N \tag{3.3}$$

and $G^{(N,\gamma)}$ is a function governing the weight transfer. We assume

$$G^{(N,\gamma)}(\bar{z}, i, j, k, l) \leqslant \begin{cases} \min(g_i, g_j) & \text{if} \quad i \neq j \\ \frac{1}{2} g_i & \text{if} \quad i=j \end{cases} \tag{3.4}$$

where $\bar{z}=(g_1,..., g_N)$ and $i, j, k, l=1,..., N$, so that the components of the process remain positive. Note that mass is preserved, i.e.,

$$\sum_{i=1}^{N} g_i^{(N,\gamma)}(t)=\sum_{i=1}^{N} g_i^{(N,\gamma)}(0), \qquad \forall t>0$$

The function $D^{(N,\gamma)}$, expressing the intensity of the jumps, is assumed to be measurable and bounded in $\bar{z}$.

The process $Z(t)$ is a jump process, which models a random mass flow. The waiting time between successive jumps has an exponential distribution with a parameter which is determined by the function $D^{(N,\gamma)}$. Each jump is characterized by random indices $i, j, k, l$. During the jump, a part of the weights $g_i, g_j$, which is determined by the funtion $G^{(N,\gamma)}$, is transferred to the weights $g_k, g_l$.

**Theorem 3.1.** Let $p_i^{(N)}(t)$, $i=1,..., N$, $t\geqslant 0$, be the solution to a discrete velocity model (1.11)–(1.13).

Let the parameters $D^{(N,\gamma)}$ and $G^{(N,\gamma)}$ of the stochastic process $Z(t)$ be related to the parameter $Q^{(N)}$ of the discrete velocity model via the condition

$$D^{(N,\gamma)}(\bar{z}, i, j, k, l)\, G^{(N,\gamma)}(\bar{z}, i, j, k, l)=Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\})\, g_i g_j$$
$$\forall \gamma\geqslant 1, \qquad \forall \bar{z}=(g_1,..., g_N), \qquad \forall i, j, k, l=1,..., N \tag{3.5}$$

Let the funtion $G^{(N,\gamma)}$ satisfy (3.4) and

$$G^{(N,\gamma)}(\bar{z}, i, j, k, l)\leqslant C_{g,\max}\frac{1}{\gamma}, \qquad \forall \gamma\geqslant 1, \quad \forall \bar{z}, \quad \forall i, j, k, l=1,..., N \tag{3.6}$$

If

$$\lim_{\gamma\to\infty} E\sum_{i=1}^{N} |p_i^{(N)}(0)-g_i^{(N,\gamma)}(0)|=0 \tag{3.7}$$

then

$$\lim_{\gamma\to\infty} E \sup_{0\leqslant t\leqslant T} \sum_{i=1}^{N} |p_i^{(N)}(t)-g_i^{(N,\gamma)}(t)|=0, \qquad \forall T>0$$

**Remark 3.2.** There are considerable degrees of freedom in the choice of the parameters $D^{(N,\gamma)}$ and $G^{(N,\gamma)}$ of the stochastic process $Z(t)$. We mention as an example the functions

$$G^{(N,\gamma)}(\bar{z}, i, j, k, l) = \frac{1}{\gamma} \frac{g_i g_j}{g_i + g_j}$$

and

$$D^{(N,\gamma)}(\bar{z}, i, j, k, l) = \gamma(g_i + g_j)\, Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\})$$

where $\bar{z} = (g_1,..., g_N)$ and $i, j, k, l = 1,..., N$. Obviously, conditions (3.4)–(3.6) are fulfilled for the above functions.

**Remark 3.3.** The dependence of the functions $D^{(N,\gamma)}$ and $G^{(N,\gamma)}$ on the parameter $\gamma$ may be rather general provided that conditions (3.4)–(3.6) are satisfied.

The effect of $\gamma$ becoming large can be described as follows. The part of the weights which is transferred during each jump decreases according to condition (3.6). On the other hand, the intensity function $D^{(N,\gamma)}$ increases according to condition (3.5), and so does the parameter of the waiting time distribution. Thus, when $\gamma$ becomes large, the mean number of jumps on a given time interval increases, while the amount of weight transfer during each jump decreases.

Consider the example given in Remark 3.2. In the particular case

$$Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \mathscr{X}^{(N)}, \mathscr{X}^{(N)}) = \text{const} = C^{(N)}$$

the parameter of the waiting time distribution is

$$\sum_{i, j, k, l=1}^{N} D^{(N,\gamma)}(\bar{z}, i, j, k, l) = \gamma \sum_{i, j=1}^{N} (g_i + g_i)\, C^{(N)} = \gamma C^{(N)} 2N\lambda_0^{(N)}(\mathscr{X}^{(N)})$$

so that the expected value of the waiting time between two jumps is proportional to $1/\gamma$.

*Proof of Theorem 3.1.* We use the following martingale representation for Markov processes (cf., e.g., ref. 4, Chapter 4, Proposition 1.7). Let $\Phi$ be a function from the domain $\mathscr{D}(A)$ of the generator $\mathscr{A}$ (i.e., an arbitrary bounded measurable function in our case). Then

$$\Phi(Z(t)) = \Phi(Z(0)) + \int_0^t \mathscr{A}(\Phi)(Z(s))\, ds + M(t) \tag{3.8}$$

where $M(t)$ is a martingale. Moreover, if $\Phi^2 \in \mathscr{D}(\mathscr{A})$, then

$$E[M(t)]^2 = E\int_0^t [\mathscr{A}(\Phi^2) - 2\Phi\mathscr{A}(\Phi)](Z(s))\, ds \tag{3.9}$$

We will apply (3.8), (3.9) to the process $Z(t)$ with the generator (3.1). Consider a function of the form

$$\Phi(\bar{z}) = \sum_{i=1}^{N} g_i \varphi_i, \qquad \bar{z} = (g_1, ..., g_N) \tag{3.10}$$

where $\varphi \in \mathscr{R}^N$ is a fixed vector. Notice that

$$\Phi(J^{(N,\gamma)}(\bar{z}, i, j, k, l)) = \Phi(\bar{z}) + G^{(N,\gamma)}(\bar{z}, i, j, k, l)[\varphi_k + \varphi_l - \varphi_i - \varphi_j] \tag{3.11}$$

according to (3.2), (3.3). Thus,

$$\begin{aligned}\mathscr{A}(\Phi)(\bar{z}) &= \sum_{i,j,k,l=1}^{N} D^{(N,\gamma)}(\bar{z}, i, j, k, l)\, G^{(N,\gamma)}(\bar{z}, i, j, k, l)[\varphi_k + \varphi_l - \varphi_i - \varphi_j] \\ &= \sum_{i,j,k,l=1}^{N} Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\})[\varphi_k + \varphi_l - \varphi_i - \varphi_j]\, g_i g_j\end{aligned} \tag{3.12}$$

according to assumption (3.5). It follows from (3.11) that

$$\begin{aligned}&\Phi^2(J^{(N,\gamma)}(\bar{z}, i, j, k, l)) \\ &\quad = \Phi^2(\bar{z}) + 2\Phi(\bar{z})\, G^{(N,\gamma)}(\bar{z}, i, j, k, l) \\ &\qquad \times [\varphi_k + \varphi_l - \varphi_i - \varphi_j] + [G^{(N,\gamma)}(\bar{z}, i, j, k, l)]^2 [\varphi_k + \varphi_l - \varphi_i - \varphi_j]^2\end{aligned}$$

and, consequently,

$$\begin{aligned}\mathscr{A}(\Phi^2)(\bar{z}) = 2\Phi(\bar{z})\, &\mathscr{A}(\Phi)(\bar{z}) \\ &+ \sum_{i,j,k,l=1}^{N} D^{(N,\gamma)}(\bar{z}, i, j, k, l)[G^{(N,\gamma)}(\bar{z}, i, j, k, l)]^2 \\ &\times [\varphi_k + \varphi_l - \varphi_i - \varphi_j]^2\end{aligned} \tag{3.13}$$

From (3.13), (3.5), and (3.6), we obtain the estimates

$$\begin{aligned}&|\mathscr{A}(\Phi^2)(\bar{z}) - 2\Phi(\bar{z})\, \mathscr{A}(\Phi)(\bar{z})| \\ &\quad \leqslant 16 \max_i |\varphi_i| \sum_{i,j,k,l=1}^{N} Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\})\, g_i g_j G^{(N,\gamma)}(\bar{z}, i, j, k, l) \\ &\quad \leqslant 16 \max_i |\varphi_i|\, C_{g,\max} \frac{1}{\gamma} C_{Q^{(N)},\max} \left(\sum_{i=1}^{N} g_i\right)^2\end{aligned}$$

and, using (3.9),

$$E[M(t)]^2 \leqslant 16 \max_i |\varphi_i| \, N^2 [C_{g,\max}]^3 \frac{1}{\gamma} C_{Q^{(N)},\max} t \tag{3.14}$$

Using the vectors $\varphi_i^{(m)} = \psi_{m,i}$, $m, i = 1,..., N$, where the symbols $\psi_{m,i}$ are defined in (3.3), one obtains from (3.8), (3.10), and (3.12)

$$\begin{aligned} g_m^{(N,\gamma)}(t) = g_m^{(N,\gamma)}(0) + \int_0^t \sum_{i,j,k,l=1}^{N} [\psi_{m,k} + \psi_{m,l} - \psi_{m,i} - \psi_{m,j}] \\ \times Q^{(N)}(\xi_i^{(N)}, \xi_j^{(N)}, \{\xi_k^{(N)}\}, \{\xi_l^{(N)}\}) \, g_i^{(N,\gamma)}(s) \, g_j^{(N,\gamma)}(s) \, ds + M_m(t) \end{aligned} \tag{3.15}$$

Comparing (3.15) with (1.9), one obtains the estimate

$$\begin{aligned} &|p_m^{(N)}(t) - g_m^{(N,\gamma)}(t)| \\ &\quad \leqslant |p_m^{(N)}(0) - g_m^{(N,\gamma)}(0)| \\ &\quad\quad + \int_0^t 4 \, C_{Q^{(N)},\max} 2NC_{g,\max} \sum_{i=1}^{N} |p_i^{(N)}(s) - g_i^{(N,\gamma)}(s)| \, ds + |M_m(t)| \end{aligned} \tag{3.16}$$

Taking the sum with respect to $m$ in (3.16) and applying Gronwall's inequality, one obtains

$$\begin{aligned} &\sum_{i=1}^{N} |p_i^{(N)}(t) - g_i^{(N,\gamma)}(t)| \\ &\quad \leqslant \exp(8C_{Q^{(N)},\max} C_{g,\max} N^2 t) \\ &\quad\quad \times \left[ \sum_{i=1}^{N} |p_i^{(N)}(0) - g_i^{(N,\gamma)}(0)| + \sum_{i=1}^{N} |M_i(t)| \right] \end{aligned}$$

After taking the supremum with respect to $t \in [0, T]$ and the mathematical expectation, the assertion of Theorem 3.1 follows from (3.14), (3.7), and the martingale inequality. ▮

## 4. CONCLUDING REMARKS

Theorem 2.1 provides a rather general solution to the problem of approximating the Boltzmann equation (1.6) by discrete velocity models. This result was possible, since we have neglected the properties of conservation of momentum and energy. These properties are fulfilled only approximately for the discrete velocity model defined in (2.1)–(2.4). The difficulties connected with the "closure problem" for discrete velocity models were discussed in detail in ref. 5. We refer to refs. 7 and 9 concerning the construction of discrete velocity models possessing conservation properties.

Theorem 3.1 provides a simple stochastic process approximating the solution to a discrete velocity model. This process can be applied for solving a discrete velocity model numerically. To this end, the exponentially distributed waiting time between the jumps should be approximated in an appropriate way. Such a procedure was discussed in some detail in ref. 13, where Bird's DSMC algorithm was treated.

The application of the numerical algorithm based on Theorem 3.1 to the spatially inhomogeneous Boltzmann equation is straightforward if one applies the usual technique of splitting the free flow and the collision simulation. However, as far as numerical applications are concerned, conservation properties of the algorithm become essential.

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